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A construction of solutions to reflection equations for interaction-round-a-face models

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Abstract. We present a procedure in which known solutions to reflection equations for interaction-round-a-face lattice models are used to construct new solutions. The procedure is particularly well suited to models which have a known fusion hierarchy and which are based on graphs containing a node of valency 1. Among such models are the Andrews–Baxter–Forrester models, for which we construct reflection equation solutions for fixed and free boundary conditions.

1. Introduction

Boundary weights which satisfy reflection equations are important in the study of solvable interaction-round-a-face (IRF) lattice models with non-periodic boundary conditions [1–11]. More specifically, such boundary weights lead to families of commuting transfer matrices and hence integrability. In [1, 3, 8–10] boundary weights for particular models were obtained by directly solving the IRF reflection equations, while in [4] they were obtained using intertwiners together with known boundary weights for a related vertex model.

Here we present a procedure in which known boundary weights for an IRF model—together with auxiliary face weights, generally obtained from a fusion hierarchy—are used to construct new boundary weights for that model. This procedure takes two forms, one which leads to weights for fixed boundary conditions and the other which leads to weights for free boundary conditions. In each case, the resulting boundary weights contain an arbitrary parameter.

Our procedure is particularly effective for models, such as the Andrews–Baxter–Forrester (ABF) models [12], which are based on graphs containing a node of valency 1, since there then exist trivial weights which can be used as the known starting weights. In this paper, we apply our procedure to the ABF models and obtain weights for fixed boundary conditions, which match those of [1], as well as weights for free boundary conditions.

2. General procedure

2.1. Motivation

We begin with a brief outline of the motivation for our procedure. Using the techniques of [1], it can be seen that families of commuting transfer matrices with non-periodic

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boundary conditions can be obtained if there exist face and boundary weights, W and B, which satisfy the Yang–Baxter equation, an inversion relation, and a reflection equation of the form

$$\sum_{f,g_0\ldots g_n} W\left(\begin{matrix} c & f \\ b & a_0 \end{matrix}\middle| u-v\right) W\left(\begin{matrix} d & g_0 \\ c & f \end{matrix}\middle| \mu-u-v\right) B\left(f \begin{matrix} g_0\ldots g_n \\ a_0\ldots a_n \end{matrix}\middle| u\right) B\left(d \begin{matrix} e_0\ldots e_n \\ g_0\ldots g_n \end{matrix}\middle| v\right)$$
$$= \sum_{f,g_0\ldots g_n} W\left(\begin{matrix} e_0 & f \\ d & c \end{matrix}\middle| u-v\right) W\left(\begin{matrix} f & g_0 \\ c & b \end{matrix}\middle| \mu-u-v\right) B\left(f \begin{matrix} e_0\ldots e_n \\ g_0\ldots g_n \end{matrix}\middle| u\right)$$
$$\times B\left(b \begin{matrix} g_0\ldots g_n \\ a_0\ldots a_n \end{matrix}\middle| v\right) \tag{2.1}$$

where μ and n are fixed, and u and v are arbitrary values of the spectral parameter.

It can be further observed that if B satisfies (2.1) for $n = 0$, then B' defined by

$$B'\left(c_0 \begin{matrix} b_0\ldots b_n \\ a_0\ldots a_n \end{matrix}\middle| u\right) = \sum_{c_1\ldots c_n} \left[\prod_{j=1}^{n} W\left(\begin{matrix} c_{j-1} & c_j \\ a_{j-1} & a_j \end{matrix}\middle| u+\xi_j\right)\right.$$
$$\left. \times W\left(\begin{matrix} b_{j-1} & b_j \\ c_{j-1} & c_j \end{matrix}\middle| \mu-u+\xi_j\right)\right] B\left(c_n \begin{matrix} b_n \\ a_n \end{matrix}\middle| u\right) \tag{2.2}$$

satisfies (2.1) for $n > 0$, where $\xi_1\ldots\xi_n$ are fixed inhomogeneities.

Finally, it can be seen that in certain cases the internal spins in B' can be eliminated using fusion projection operators to give boundary weights B'' which are different from B, but which again satisfy (2.1) for $n = 0$. This assumes that B has the form $B\left(c \begin{matrix} b \\ a \end{matrix}\middle| u\right) = \bar{B}(c\ \ a\ |u)\delta_{ab}$, and that the inhomogeneities are given by $\xi_j = \xi - (n-j)\lambda$, where ξ is arbitrary and λ is the crossing parameter.

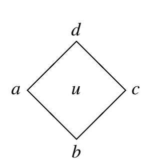
2.2. Adjacency condition and face weights

We now proceed to a detailed presentation of our procedure. We are considering an IRF model on a square lattice, and we assume that there are restrictions on the spins allowed on any adjacent lattice sites, as specified by an adjacency matrix

$$A_{ab} = \begin{cases} 0 & \text{spins } a \text{ and } b \text{ may not be adjacent} \\ 1 & \text{spins } a \text{ and } b \text{ may be adjacent.} \end{cases}$$

For such models, we associate a Boltzmann weight W with each set of spins a, b, c and d that are allowed to be adjacent around a face, i.e. for which $A_{ab}A_{bc}A_{cd}A_{da} = 1$. These weights are denoted

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right) \quad = \tag{2.3}$$

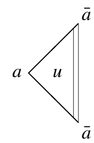


where u is the spectral parameter.

2.3. Fixed boundary conditions

We now consider a boundary containing a fixed spin $\bar{a}$. In this case we associate a boundary weight with each spin a which is allowed to be adjacent to $\bar{a}$,

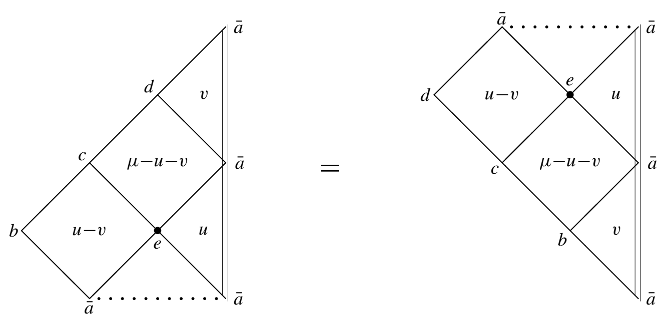
$$\bar{B}(a \;\; \bar{a} \;|u) \quad = \tag{2.4}$$



Such boundary weights generally lead only to quasi-fixed boundary conditions, since although $\bar{a}$ is fixed, there is still a weighted sum over every spin adjacent to $\bar{a}$. However, there will be genuine fixed boundary conditions at any value of the spectral parameter for which only one of the boundary weights is non-zero.

The boundary weights (2.4), together with the face weights (2.3), are expected to satisfy the fixed-boundary reflection equations for $\bar{a}$. There is one such equation for each set of spins b, c and d satisfying $A_{\bar{a}b}A_{bc}A_{cd}A_{d\bar{a}} = 1$,

$$\sum_e W\left(\begin{matrix} c & e \\ b & \bar{a} \end{matrix}\middle| u-v\right) W\left(\begin{matrix} d & \bar{a} \\ c & e \end{matrix}\middle| \mu-u-v\right) \bar{B}(e \;\; \bar{a} \;|u)\bar{B}(d \;\; \bar{a} \;|v)$$
$$= \sum_e W\left(\begin{matrix} \bar{a} & e \\ d & c \end{matrix}\middle| u-v\right) W\left(\begin{matrix} e & \bar{a} \\ c & b \end{matrix}\middle| \mu-u-v\right) \bar{B}(e \;\; \bar{a} \;|u)\bar{B}(b \;\; \bar{a} \;|v). \tag{2.5}$$



Here, μ is a fixed parameter and the sums are over all spins e satisfying $A_{\bar{a}e}A_{ec} = 1$. We note that if the face weights satisfy the symmetry

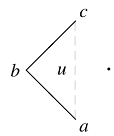
$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right) = W\left(\begin{matrix} b & c \\ a & d \end{matrix}\middle| u\right) \tag{2.6}$$

then (2.5) is automatically satisfied whenever $b = d$. Furthermore, in the case in which there is only one spin a allowed to be adjacent to $\bar{a}$, i.e. $\bar{a}$ has a valency of 1, we must have $b = d = e = a$ in (2.5) implying that the equation is always satisfied, that the single boundary weight $\bar{B}(a \;\; \bar{a} \;|u)$ may be assigned to any function of u, and that we have genuine fixed boundary conditions for all u.

2.4. Free boundary conditions

For the case of free boundary conditions, we associate a boundary weight with each set of spins a, b and c satisfying $A_{ab}A_{bc} = 1$,

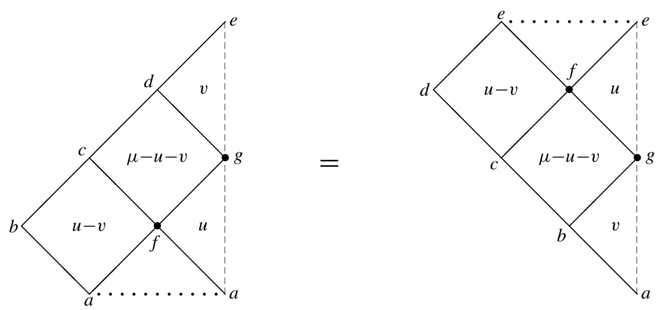
$$B\left(b\ \begin{matrix} c \\ a \end{matrix}\middle|\, u\right) \quad = \tag{2.7}$$



Such boundary weights generally lead only to quasi-free boundary conditions, in the sense that there is a weighted sum over the boundary spins. However, there will be genuine free boundary conditions at any value of the spectral parameter for which all of the boundary weights are equal and non-zero.

The boundary weights (2.7), together with the face weights (2.3), are expected to satisfy the free-boundary reflection equations. There is one such equation for each set of spins a, b, c, d and e satisfying $A_{ab}A_{bc}A_{cd}A_{de} = 1$,

$$\sum_{fg} W\left(\begin{matrix} c & f \\ b & a \end{matrix}\middle|\, u-v\right) W\left(\begin{matrix} d & g \\ c & f \end{matrix}\middle|\, \mu-u-v\right) B\left(f\ \begin{matrix} g \\ a \end{matrix}\middle|\, u\right) B\left(d\ \begin{matrix} e \\ g \end{matrix}\middle|\, v\right)$$
$$= \sum_{fg} W\left(\begin{matrix} e & f \\ d & c \end{matrix}\middle|\, u-v\right) W\left(\begin{matrix} f & g \\ c & b \end{matrix}\middle|\, \mu-u-v\right) B\left(f\ \begin{matrix} e \\ g \end{matrix}\middle|\, u\right) B\left(b\ \begin{matrix} g \\ a \end{matrix}\middle|\, v\right). \tag{2.8}$$

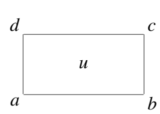


Here, the sum on the left-hand side is over all spins f and g satisfying $A_{af}A_{cf}A_{fg}A_{gd} = 1$ and that on the right-hand side is over all spins f and g satisfying $A_{ef}A_{cf}A_{fg}A_{gb} = 1$. We note that (2.5) can be regarded as a special case of (2.8) for boundary weights of the form $B\left(b\ \begin{matrix} c \\ a \end{matrix}\middle|\, u\right) = \bar{B}(b\ \ \bar{a}\ \,|u)\delta_{a\bar{a}}\delta_{c\bar{a}}$.

2.5. Construction of new boundary weights

Our construction of new boundary weights requires that there exist an auxiliary adjacency matrix $\bar{A}$ and, for each set of spins a, b, c and d satisfying $\bar{A}_{ab}A_{bc}\bar{A}_{cd}A_{da} = 1$, an auxiliary face weight

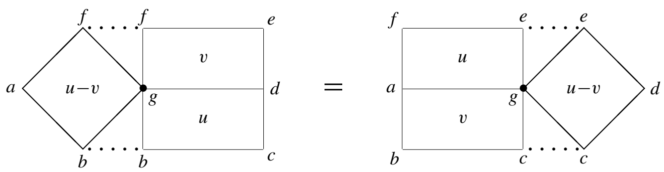
$$\bar{W}\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle|\, u\right) \quad = \tag{2.9}$$



These weights, together with the fundamental face weights (2.3), are assumed to satisfy the auxiliary Yang–Baxter equations. There is one such equation for each set of spins a, b, c, d, e and f satisfying $A_{ab}\bar{A}_{bc}A_{cd}A_{de}\bar{A}_{ef}A_{fa} = 1$,

$$\sum_g W\left(\begin{matrix} f & g \\ a & b \end{matrix}\middle| u-v\right) \bar{W}\left(\begin{matrix} g & d \\ b & c \end{matrix}\middle| u\right) \bar{W}\left(\begin{matrix} f & e \\ g & d \end{matrix}\middle| v\right)$$

$$= \sum_g \bar{W}\left(\begin{matrix} a & g \\ b & c \end{matrix}\middle| v\right) \bar{W}\left(\begin{matrix} f & e \\ a & g \end{matrix}\middle| u\right) W\left(\begin{matrix} e & d \\ g & c \end{matrix}\middle| u-v\right). \tag{2.10}$$

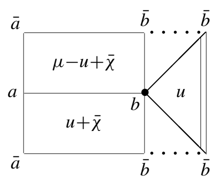


Here, the sum on the left-hand side is over all spins g satisfying $A_{bg}A_{fg}\bar{A}_{gd} = 1$ and that on the right-hand side is over all spins g satisfying $\bar{A}_{ag}A_{gc}A_{ge} = 1$.

In practice, such auxiliary face weights can often be constructed from a row of fundamental face weights with appropriate inhomogeneities, using fusion projection operators to eliminate internal spins.

Our construction of new boundary weights takes two forms. In the first form, we obtain new weights for a boundary with fixed spin $\bar{a}$ using known weights for a boundary with fixed spin $\bar{b}$, where we assume that, with respect to $\bar{A}$, $\bar{a}$ is the only spin allowed to be adjacent to $\bar{b}$. The new weights depend on an arbitrary parameter $\bar{\chi}$ and, for each spin a allowed to be adjacent to $\bar{a}$, are defined as

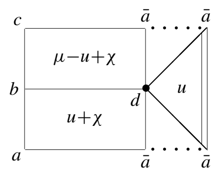
$$\bar{B}'(a \;\; \bar{a} \;|u) = \sum_b \bar{W}\left(\begin{matrix} a & b \\ \bar{a} & \bar{b} \end{matrix}\middle| u+\bar{\chi}\right) \bar{W}\left(\begin{matrix} \bar{a} & \bar{b} \\ a & b \end{matrix}\middle| \mu-u+\bar{\chi}\right) \bar{B}(b \;\; \bar{b} \;|u) \tag{2.11}$$

$=$ 

where the sum is over all spins b satisfying $\bar{A}_{ab}A_{b\bar{b}} = 1$. We note that we suppress the dependence of these weights on the spin $\bar{b}$ and the parameter $\bar{\chi}$. It is straightforward to show that the new weights (2.11) satisfy the fixed-boundary reflection equations for $\bar{a}$, using the assumptions that the known weights satisfy the fixed-boundary reflection equations for $\bar{b}$, that the auxiliary face weights satisfy (2.10), and that $\bar{b}$ has valency 1 with respect to $\bar{A}$.

In the second form of our construction of new boundary weights, we obtain certain weights for free boundary conditions using known weights for a boundary with fixed spin $\bar{a}$. The new weights depend on an arbitrary parameter χ and, for each set of spins a, b and c satisfying $\bar{A}_{\bar{a}a}A_{ab}A_{bc}\bar{A}_{c\bar{a}} = 1$, are defined as

$$B\left(b \;\begin{matrix} c \\ a \end{matrix}\middle| u\right) = \sum_d \bar{W}\left(\begin{matrix} b & d \\ a & \bar{a} \end{matrix}\middle| u+\chi\right) \bar{W}\left(\begin{matrix} c & \bar{a} \\ b & d \end{matrix}\middle| \mu-u+\chi\right) \bar{B}(d \;\; \bar{a} \;|u) \tag{2.12}$$

$=$ 

where the sum is over all spins d satisfying $\bar{A}_{bd}A_{d\bar{a}} = 1$. Again we suppress the dependence of these weights on the spin $\bar{a}$ and the parameter χ. The new weights (2.12) satisfy the free-boundary reflection equations for each set of spins a, b, c, d and e in (2.8) which satisfy $\bar{A}_{\bar{a}a}A_{ab}A_{bc}A_{cd}A_{de}\bar{A}_{e\bar{a}} = 1$. This follows straightforwardly from the assumptions that the known weights satisfy the fixed-boundary reflection equations for $\bar{a}$, and that the auxiliary face weights satisfy (2.10).

3. ABF models

3.1. Adjacency condition and face weights

We now consider the Andrews–Baxter–Forrester (ABF) models [12]. There is one such model for each integer $L \geqslant 3$, with the spins a in this model taking the values

$$a \in \{1, 2, \ldots, L\}. \tag{3.1}$$

The adjacency matrix is defined by the condition that $A_{ab} = 1$ if and only if

$$|a - b| = 1. \tag{3.2}$$

The face weights are given by

$$\begin{aligned}
W\left(\begin{matrix} a\pm 1 & a \\ a & a\mp 1 \end{matrix}\middle| u\right) &= \frac{\theta(\lambda - u)}{\theta(\lambda)} \\
W\left(\begin{matrix} a & a\pm 1 \\ a\mp 1 & a \end{matrix}\middle| u\right) &= \sqrt{\frac{\theta((a-1)\lambda)\,\theta((a+1)\lambda)}{\theta(a\lambda)^2}}\,\frac{\theta(u)}{\theta(\lambda)} \\
W\left(\begin{matrix} a & a\pm 1 \\ a\pm 1 & a \end{matrix}\middle| u\right) &= \frac{\theta(a\lambda \pm u)}{\theta(a\lambda)}
\end{aligned} \tag{3.3}$$

where θ is the elliptic theta-1 function of fixed nome and

$$\lambda = \frac{\pi}{L+1} \tag{3.4}$$

is the crossing parameter. We note that when constructing boundary weights it will be convenient to make the choice

$$\mu = \lambda. \tag{3.5}$$

For the ABF models, an auxiliary adjacency matrix and auxiliary face weights which satisfy (2.10) are provided by the level n fused adjacency matrix and the n by 1 fused face weights [13–15]

$$\bar{A} = A^n \qquad \overline{W} = W^{n,1} \tag{3.6}$$

where

$$n \in \{0, 1, \ldots, L-1\}. \tag{3.7}$$

The level n fused adjacency matrix is defined by the condition that $A^n_{ab} = 1$ if and only if

$$a - b \in \{-n, -n+2, \ldots, n-2, n\} \tag{3.8}$$

and

$$a + b \in \{n+2, n+4, \ldots, 2L-n-2, 2L-n\}. \tag{3.9}$$

We note that $A^1 = A$. The n by 1 fused face weights are defined in terms of rows of n fundamental face weights (3.3) and, after appropriate normalization and symmetrization, are given by

$$W^{n,1}\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right) \tag{3.10}$$

$$= \begin{cases} \epsilon_b\,\epsilon_d \sqrt{\dfrac{\theta((a+b\mp n)\lambda/2)\theta((c+d\pm n)\lambda/2)}{\theta(b\lambda)\theta(d\lambda)}} \\ \qquad \times \dfrac{\theta(u-(n\pm(a-b))\lambda/2)}{\theta(\lambda)} & c = b\pm 1, d = a\pm 1 \\ \epsilon_b\,\epsilon_d \sqrt{\dfrac{\theta((n\mp(a-b))\lambda/2)\theta((n\pm(d-c))\lambda/2)}{\theta(b\lambda)\theta(d\lambda)}} \\ \qquad \times \dfrac{\theta((a+b\pm n)\lambda/2\mp u)}{\theta(\lambda)} & c = b\mp 1, d = a\pm 1 \end{cases}$$

where ϵ_a are factors whose required properties are

$$(\epsilon_a)^2 = 1 \qquad \epsilon_a\epsilon_{a+2} = -1. \tag{3.11}$$

We note that the fused weights (3.10) reduce to (3.3) for $n = 1$.

3.2. Weights for fixed boundary conditions

Since, for the ABF models, the spin 1 has valency 1 with respect to A, and the face weights satisfy the symmetry (2.6), the boundary weight $\bar{B}(2\ \ 1\,|u)$ can be set to an arbitrary function of u. Furthermore, it follows from (3.8) and (3.9) that the spin 1 has valency 1 with respect to any $A^{\bar{a}-1}$, the only allowed neighbour being the spin $\bar{a}$. It is therefore possible to construct new weights for a boundary with fixed spin $\bar{a}$ using an arbitrary weight for a boundary with fixed spin 1. Accordingly, we apply (2.11) with $\bar{A} = A^{\bar{a}-1}$, $\overline{W} = W^{\bar{a}-1,1}$, $\bar{b} = 1$, $\mu = \lambda$, $\bar{\chi} = -\lambda - \bar{\xi}$, and $\bar{B}(2\,1|u) = \epsilon_1\,\epsilon_2\,\epsilon_{\bar{a}}\,\epsilon_{\bar{a}-1}\sqrt{\theta(2\lambda)/\theta(\lambda)}\,g(u)$, which gives

$$\bar{B}'(\bar{a}\pm 1\ \ \bar{a}|u) = g(u)\sqrt{\frac{\theta((\bar{a}\pm 1)\lambda)}{\theta(\bar{a}\lambda)}}\,\frac{\theta(u\pm\bar{\xi})\,\theta(u\mp\bar{a}\lambda\mp\bar{\xi})}{\theta(\lambda)^2} \tag{3.12}$$

where $\bar{\xi}$ is an arbitrary constant and g is an arbitrary function. It can be seen that these weights exactly match those obtained in [1] by directly solving the reflection equations, and that there exist values of u, such as $u = \pm\bar{\xi}$, at which we have genuine fixed boundary conditions.

3.3. Weights for free boundary conditions

We now consider the construction of ABF weights for free boundary conditions using (2.12) together with (3.6) and (3.12). We shall associate with any ABF weight $B\left(b\ \begin{matrix} c \\ a \end{matrix}\middle| u\right)$ either odd or even parity, according to the parity of b. Due to (3.2), each free-boundary reflection equation (2.8) contains boundary weights all with the same parity. Similarly, due to (3.8) and (3.9), (2.12) generates boundary weights all with the same parity. The requirement in (2.12) that we have $A^n_{a\bar{a}}A^n_{c\bar{a}} = 1$ also implies that, in general, there might not be a weight $B\left(b \begin{smallmatrix} c \\ a \end{smallmatrix}\middle| u\right)$ generated for each b of the appropriate parity. However, by examining (3.8) and (3.9), we find that the values

$$(n,\bar{a}) = \begin{cases} \left(\frac{L-2}{2}, \frac{L}{2}\right) \text{ or } \left(\frac{L}{2}, \frac{L+2}{2}\right); & L \text{ even, weights even} \\ \left(\frac{L-2}{2}, \frac{L+2}{2}\right) \text{ or } \left(\frac{L}{2}, \frac{L}{2}\right); & L \text{ even, weights odd} \\ \left(\frac{L-1}{2}, \frac{L+1}{2}\right); & L \text{ odd, weights even} \\ \left(\frac{L-3}{2}, \frac{L+1}{2}\right), \left(\frac{L-1}{2}, \frac{L-1}{2}\right), & \\ \left(\frac{L-1}{2}, \frac{L+3}{2}\right) \text{ or } \left(\frac{L+1}{2}, \frac{L+1}{2}\right); & L \text{ odd, weights odd} \end{cases} \tag{3.13}$$

do generate a full set of boundary weights of a given parity. We now apply (2.12) with $\mu = \lambda$, $\chi = \xi + (n-1)\lambda/2$, $g(u) \mapsto \epsilon_{\bar{a}}\,\epsilon_{\bar{a}-1}\,g(u)$, and $\bar{\xi} \mapsto \bar{\xi} - \bar{a}\lambda/2$, which gives

$$\begin{aligned} B\left(a \begin{smallmatrix} a\pm 1 \\ a\mp 1 \end{smallmatrix}\middle| u\right) = g(u)&\sqrt{\frac{\theta(a\lambda)}{\theta((a\pm 1)\lambda)}} \\ &\times\{\theta((n+1-a+\bar{a})\lambda/2)\theta((n+1+a-\bar{a})\lambda/2)\theta((a+\bar{a}-n-1)\lambda/2) \\ &\times\theta((a+\bar{a}+n+1)\lambda/2)[\theta(a\lambda)\theta(\lambda)]^{-2}\}^{1/2} \\ &\times\frac{\theta(a\lambda/2-\bar{\xi}\mp\xi)\theta(a\lambda/2+\bar{\xi}\mp\xi)}{\theta(\lambda)^2}\,\frac{\theta(2u)}{\theta(\lambda)} \end{aligned} \tag{3.14}$$

$$\begin{aligned} B\left(a \begin{smallmatrix} a\pm 1 \\ a\pm 1 \end{smallmatrix}\middle| \pm u\right) = g(u)&\sqrt{\frac{\theta(a\lambda)}{\theta((a\pm 1)\lambda)}}\{\theta((a+\bar{a}-n-1)\lambda/2)\theta((a+\bar{a}+n+1)\lambda/2) \\ &\times\theta(u+\bar{a}\lambda/2-\bar{\xi})\theta(u+\bar{a}\lambda/2+\bar{\xi})\theta(u+a\lambda-\bar{a}/2-\xi) \\ &\times\theta(u+(a-\bar{a})\lambda/2+\xi) \\ &-\theta((a-\bar{a}-n-1)\lambda/2)\theta((a-\bar{a}+n+1)\lambda/2)\theta(u-\bar{a}\lambda/2-\bar{\xi}) \\ &\times\theta(u-\bar{a}\lambda/2+\bar{\xi})\theta(u+(a+\bar{a})\lambda/2-\xi)\theta(u+(a+\bar{a})\lambda/2+\xi)\} \\ &\times[\theta(a\lambda)\theta(\bar{a}\lambda)\theta(\lambda)^4]^{-1}. \end{aligned} \tag{3.15}$$

Here, the two terms which led to (3.14) were combined using a standard elliptic identity, and a common factor $\epsilon_a\,\epsilon_{a-1}$ in (3.14) and (3.15) was eliminated since, for a given $\bar{a}$, the allowed values of a must all have the same parity implying that this factor always produces the same sign.

4. Discussion

We have presented a general procedure for obtaining boundary weights for IRF models and have applied this to the ABF models. Our method should be useful for determining classes of IRF models for which solutions of the reflection equations exist and contain arbitrary parameters. For example, our method implies the existence of such solutions for the standard A–D–E models, of which the ABF models form the A series, since these are all based on graphs containing a node of valency 1 and have known fusion hierarchies.

However, we note that for the D and E series, the fused weights depend on certain internal spins so that (2.10)–(2.12) need to be generalized to include these.

In a future publication, we shall outline the relationship between the ABF weights for free boundary conditions found here and those obtained by directly solving the reflection equations or by using intertwiners. We also hope to show that these weights can be used to obtain genuine free boundary conditions at particular values of the spectral parameter and we hope to be able to show that the associated transfer matrices satisfy functional equations with the same form as in the case of fixed and periodic boundary conditions.

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